ON THE INFLUENCE OF VISCOSITY ON THE STABILITY OF EQUILIBRIUM AND STEADY-STATE ROTATION OF A RIGID BODY WITH A CAVITY, PARTIALLY FILLED WITH A VISCOUS LIQUID

(O VLIIANII VIAZKOSTI NA USTOICHIVOST' RAVNOVESII I STATSIONARNYKH VRASHCHENII TVERDOGO TELA S POLOST'IU, CHASTICHNO ZAPOLNENNOI VIAZKOI ZHIDKOST'IU)

PMM Vol.28, № 1, 1964, pp.60-67

G.K. POZHARITSKII
(Moscow)

(Received November 16, 1962)

Intuition suggests that if the energy of a system is dissipated in any motion and if, under equilibrium or steady-state motion, it has an isolated minimum, then the oscillations of the system about the undisturbed motion will damp out.

For a system with a finite number of degrees of freedom this fact can be deduced from the results of the paper by E.A. Barbashin and Krasovskii [1]. Some verifiable criteria for the dissipation of energy in any motion are suggested in [2]. An extension of the results of Barbashin and Krasovskii to systems with infinite degrees of freedom is difficult. In the present paper we make a number of assumptions on the continuity of the perturbed surface, on the velocity and on the total energy of the system, and therefore all the conclusions on the stability which we succeed in obtaining are rather conditional in nature. The mentioned restrictions are dictated by the method of the proof; it is scarcely possible that we could make further definite conclusions on the stability from the consideration of only the total energy and its derivative.

1. Let us consider a rigid body with a cavity partially filled with a viscous, incompressible liquid, subject to the action of external forces, with potential energy $V_1(q_1, q_2, \ldots)$. The ideal relations, imposed on the body as well as on its coordinates q $(i \leqslant 6)$, are assumed to be holonomic and steady-state, while the potential energy of the liquid element is taken in the form $\sigma W(X,Y,Z)d\tau$, where XYZ is the fixed system of rectangular coordinates and σ is the liquid density.

If the system is in equilibrium and if V, the potential energy of the system, has a minimum at the equilibrium position, then the equilibrium is stable in the sense of Liapunov [3,4]; in other words, for any arbitrarily small $h, \sigma', n, \varphi(N)$, we can find h_0, σ_0', n_0 such that, when $t = 0$ the initial values of $r_0^2 = q_{10}^2 + \ldots + q_{n0}^2$, of the kinetic energy T_0, of the separation N_0, and of the deviation Δ_0, satisfy the inequalities

$$r_0 < h_0, \quad T_0 < \sigma_0', \quad N_0 < n_0, \quad \Delta_0 \geqslant \varphi(N_B) \tag{1.1}$$

then, the inequalities

$$r < h, \quad T < \sigma', \quad N < n, \quad \Delta \geqslant \varphi(N) \tag{1.2}$$

will be satisfied during the whole motion or, at least, until the inequality $\Delta \geqslant \varphi(N)$, is disturbed, where $\varphi(N)$ is some possible deviation. Let us assume that it is not disturbed, i.e. all the latter inequalities are satisfied during the whole motion.

The derivative of the total energy E with respect to time satisfies Equation

$$\frac{dE}{dt} = -\int_D \Phi \, d\tau$$

under the condition that the relative velocity of the liquid equals zero on the wetted surface of the cavity, while the stress on the free boundary is perpendicular to it and constant.

Here Φ is the dissipation function of the viscus liquid, taken in the Navier-Stokes form, and D is the region occupied by the liquid. The set of states of the system which satisfy inequalities (1.1) will be called the region H_0, and for inequalities (1.2), the region H.

Let $\varphi(x, y, z, t) = 0$ be the equation of the free surface in the moving system of coordinates xyz, connected to the rigid body and coinciding with the fixed system at the equilibrium position. Also, let the vector $\mathbf{V}(x, y, z, t)$ represent the field of velocity of the liquid relative to the body, with components u, v, w along the axes x, y, z and, under the given initial conditions $\varphi_0(x, y, z, 0) = 0$, $\mathbf{V}(x, y, z, 0)$, q_{i0}, $\dot{q}_{i0}$ let the subsequent motion be determined uniquely.

The set of functions $\varphi(x, y, z, t) = 0$, u, v, w and the quantities $q_i, \dot{q}_i$, will be called a state of the system and will be denoted by M; the initial state will be denoted by M_0, and the initial state corresponding to a null field of the relative velocities, by M_{00}. The relative kinetic energy of the liquid will be denoted by T_r. Let us consider the equilibrium surface $W(X, Y, Z) = \alpha_0$ and in its neighborhood let us introduce the curvilinear coordinates by means of the substitutions

$$\lambda = \lambda(x, y, z), \qquad \nu = \nu(x, y, z), \qquad \tau = W(x, y, z)$$

continuous in the neighborhood of $W(x, y, z) = \alpha_0$ and allowing of continuous inverses. Moreover, let the equation of the wall of cavity S be $\nu(x, y, z) = \beta_0 = \text{const}$, and let the equation of any free surface be representable in the form $\tau - \alpha_0 = \varkappa(\lambda, \nu, t)$. Let us introduce two systems of assumptions.

Assumption 1.1 will consist of two parts.

1.1.1) The function $\varkappa(\lambda,\nu,t)$ is uniformly continuous in λ,ν for all $t>0$; i.e. for any δ, M_0 we can find an $\varepsilon(\delta, M_0)$ such that from the inequality

$$(\lambda-\lambda')^2+(\nu-\nu')^2<\varepsilon(\delta, M_0)$$

will follow the inequality $\varkappa(\lambda,\nu,t)-\varkappa(\lambda',\nu',t)|<\delta$ for all possible pairs $(\lambda, \nu),(\lambda', \nu')$ in the region $[W(x,y,z)=\alpha_0$, inside $S]$ and for all $t>0$.

1.1.2) For any δ, t^*, M_{00} we can find such a constant $\gamma(\delta, t^*, M_{00})$ that if at the initial instant the inequalities

$$|\varkappa(\lambda,\nu,0)-\varkappa'(\lambda,\nu,0)|<\gamma$$

$$K=\sum[(q_{i_0}-q_{i0}')^2+(q_{i0}^{\cdot}-q_{i0}^{\cdot\prime})^2]+T_r'+\int\limits_{D_0'}\Phi'\,d\tau<\gamma$$

are satisfied (where $\varkappa, q_{i0}, \dot q_{i0}$ correspond to state M_{00} and $\varkappa', q_{i0}', \dot q_{i0}', \ldots$, to state M_0'), then when $t=t^*$ the inequality

$$|E(M_{00},t^*)-E(M_0',t^*)|<\delta \tag{1.3}$$

is satisfied.

Assumption 1.2 also will be composed of two parts. 1.2.1). The function $\varkappa(\lambda,\nu,t)$, as well as the components of the vector $\mathbf{V}(x,y,z,t)$, are uniformly continuous.

1.2.2) We assume that inequality (1.3) follows from the inequality

$$\mathbf{V}'^2(x,y,z,0)<\gamma, \qquad |\varkappa(\lambda,\nu,0)-\varkappa'(\lambda,\nu,0)|<\gamma, \qquad K<\gamma$$

Here $\mathbf{V}'(x,y,z,0)$ is the velocity field corresponding to the state M_0'.

T h e o r e m 1.1. Let the potential energy of the system have a minimum at the equilibrium position, and let the region H not contain the motion of the liquid and the rigid body as a whole $\mathbf{V}\equiv 0$, and let assumption 1.1 be fulfilled; then, $T+r^2+N^2\to 0$ as $t\to\infty$. If instead of assumption 1.1, assumption 1.2 is fulfilled, then $T+r^2+N^2\to 0$ as $t\to\infty$ and, moreover, $\mathbf{V}^2(x,y,z,t)\to 0$ as $t\to\infty$.

P r o o f . Let us assume to the contrary that while the conditions of the theorem and also the assumption 1.1 are fulfilled, nevertheless $\lim E=E^*>0$ as $t\to\infty$. The integral

$$\int\limits_0^\infty\int\limits_D\Phi\,d\tau\,dt$$

converges; consequently, we can find such a sequence of instants $t_1,\ldots, t_k,\ldots$, that the sequence

$$\int\limits_{D_k}\Phi_k\,d\tau\to 0 \qquad \text{for } t\to\infty$$

By setting the usual condition $\mathbf{V} = 0$ on the boundary, on the basis of a well-known integral inequality [5] we can establish that the sequence $T_r(t_k) \to 0$ as $t \to \infty$

Let us consider the sequence of "points" μ_k with "coordinates"

$$\left[q_i(t_k), q_i^{\cdot}(t_k), \tau - \alpha_0 = \varkappa(\lambda, \nu, t_k), T_r(t_k), \int_{D_k} \Phi_k \, d\tau\right]$$

Since the sequence $\varkappa(\lambda, \nu, t_k)$ is uniformly bounded ($N < n$) and uniformly continuous, then by Arzel's theorem we can choose a subsequence μ_1 converging to the point μ^* with coordinates $[q_i^*, q_i^{\cdot *}, \tau - \alpha_0 = \varkappa^*(\lambda, \nu), 0, 0]$. By virtue of the continuous dependence of the total energy on the coordinates of the point μ_1 it follows that $E(\mu^*) = E^*$.

Let us choose the initial conditions at the point μ^*. Since by assumption the motion along which $\Phi \equiv 0$ does not exist, we can find a t^* such that $E(\mu^*, t^*) < E^*$. By virtue of assumption 1.1.2, for any arbitrarily small δ we can find a number $L(\delta)$ such that for all $l > L$ the inequality $|E(\mu_l, t^*) - E(\mu^*, t^*)| < \delta$, is satisfied, but this contradicts the assumption that $\lim E = E^*$ as $t \to \infty$.

The proof under assumption 1.2 differs from the one given only in that the point μ_k will have the coordinates

$$\left(q_i, q_i^{\cdot}, \tau - \alpha_0 \quad \varkappa(\lambda, \nu, t_k), u(x, y, z, t_k), v, w, \int_{D_k} \Phi(t_k) \, d\tau\right)$$

From the uniform continuity of the components of vector $\mathbf{V}$ and from the boundedness of T_r, it is easy to prove the boundedness of u, v, w. Consequently, we can choose a convergent sequence of functions u_1, v_1, w_1 and, moreover, they will necessarily converge to zero. Further the proof is analogous.

Thus, when assumption 1.1 or 1.2 and the rest of the conditions of Theorem are fulfilled, the total energy $E \to 0$ as $t \to \infty$, but hence it follows that $T + T_r + r^2 + \Delta^2 \to 0$ as $t \to \infty$. From the condition $\Delta \geqslant \varphi(N)$ it follows that $\varphi(N) \to 0$ as $t \to \infty$ and with it also $N \to 0$ as $t \to \infty$. Consequently, $T + r^2 + N^2 \to 0$ as $t \to \infty$.

Now let assumption 1.2 be fulfilled, and we can show that here the velocity field $\mathbf{V}^2(x, y, z, t) \to 0$ as $t \to \infty$.

By virtue of assumption 1.2, for any δ, M_0 we can find a $\gamma(\delta, M_0)$ such that from the inequality $(x - x')^2 + (y - y')^2 + (z - z')^2 < \gamma^2$ follows $|\mathbf{V}^2(x, y, z, t) - \mathbf{V}^2(x', y', z', t)| < \delta$ for all $t > 0$ and for all points $(x, y, z), (x', y', z')$, lying inside the liquid. If $\mathbf{V}^2(x, y, z, t)$ does not vanish when $t \to \infty$, then we can find a constant $\epsilon(M_0)$ such that for any $A > 0$ we can find a point x^*, y^*, z^* and an instant $t^* > A$ such that

$$\mathbf{V}^2(x^*, y^*, z^*, t^*) > \varepsilon.$$

Let us take $\delta < \frac{1}{2}\varepsilon$ and find A such that when $t > A$ the separation N does not exceed $\frac{1}{10}\varepsilon$. Since the distance from the "frozen" surface $W(x, y, z) = \alpha_0$, can be taken as the separation, let us do so and let us construct two surfaces $W = C$ the distance of whose points from the surface $W = \alpha_0$ does not exceed $\frac{1}{5}\gamma$. Let them have Equations

$$W(x, y, z) = \alpha_0 + \Delta\alpha, \; W(x, y, z) = \alpha_0 - \Delta\alpha.$$

If the surface $W = \alpha_0$ intersects the cavity S at a non-zero angle, then the lower bound of the volume cut out by a small sphere of radius γ (when its center is shifted along the "upper" surface $W = \alpha_0 + \Delta\alpha$) from the region $[W < \alpha_0 - \Delta\alpha$, inside $S]$, will differ from zero. Let us denote it by m'. Hence it follows that the liquid mass which has been scooped up by the small sphere of radius γ will obviously not be less than $\sigma m'$ for any $t > A$.

This means that inside the sphere

$$(x^* - x')^2 + (y^* - y')^2 + (z^* - z')^2 < \gamma^2$$

the inequality $\mathbf{V}^2(x', y', z', t^*) > {}^1/_2\varepsilon$, is satisfied and the relative kinetic energy of the mass included in this sphere, $T_\varepsilon > {}^1/_2\varepsilon\sigma m'$, which contradicts the relation $T_r \geqslant T_\varepsilon \to 0$ as $t \to \infty$.

This contradiction proves that $\mathbf{V}^2(x, y, z, t) \to 0$ as $t \to \infty$.

The assertion which is the converse of Theorem 1.1 is also valid.

T h e o r e m 1.2. If V can take negative values in an arbitrarily small neighborhood of the equilibrium position, if the region H does not contain the motion of the rigid body and the fluid as a whole, and if one of the assumptions 1.1 or 1.2 is fulfilled, then the equilibrium is unstable.

P r o o f . The proof is analogous. Let $E_0 < 0$ and let there exist the limit $E^* < 0$, but all the time let the state M lie in region H. By a similar reasoning we get a contradiction.

2. An analogous problem is encountered if $W = W(X^2 + Y^2, Z)$, and the change in q_n in a cyclic coordinate system leads to a rotation of the system as a rigid body around axis Z.

Let J be the moment of inertia of the system around axis Z and let V be the potential energy of the system, while k_0 is the angular momentum of the system around this axis. The steady-state rotation is found from Equation

$$\frac{\partial V}{\partial q_i} - \frac{k_0^2}{2J^2}\frac{\partial J}{\partial q_i} = 0 \tag{2.1}$$

under the condition that the free surface of the liquid has Equation

$$F_1 = W(X^2 + Y^2, Z) - \frac{k_0^2}{2J^2}(X^2 + Y^2) = \alpha_0 = \mathrm{const} \tag{2.2}$$

Let the steady-state motion correspond to zero values of the coordinates $q_1, \ldots, q_{n-1}$, and let the free surface $F_1 = \alpha_0$ under steady-state motion inersect the surface of the cavity S such that the normal $n_1(m)$ to the surface $F_1 = \alpha_0$ at the point m of the line of intersection with the cavity,and the normal $n_2(m)$ to the surface S everywhere from the angle $\theta(m)$, lying within the limits $\pi > \theta_2 > \theta(m) > \theta_1 > 0$. Also, let the normals n_1 and n_2 be continous and let the value of the constant α_0 not be an extreme of all the values which can be taken by F_1 in the neighborhood of $F_1 = \alpha_0$. Under these assumptions we can show [6] that the potential energy of the system and its moment of inertia around the axis Z under the condition that its liquid surface refers to the set (2.2), are single-valued functions of $q_1, \ldots, q_{n-1}$ which are continuously differentiable up to the second order for any k^2 sufficiently close to the value k_0^2 on the steady-state motion. Let us denote these functions by V'', J''.

In [6] it was shown that the functional of the measurement of the potential energy

$$\Pi = V + k_0^2 / J$$

will have a minimum on the steady-state motion if the quadratic form

$$\delta^2\Pi'' = \sum_{ij=1}^{n-1} q_i q_j \frac{\partial^2}{\partial q_i \partial q_j}\left[V'' + \frac{k_0^2}{J''}\right] \qquad (2.3)$$

(where the derivatives are computed for zero values of the coordinates) is a positive-definite form, and that the functional Π can take negative values if such values can be taken by the quadratic form. Let it be positive-definite, then its discriminant is greater than zero and in the neighborhood of zero there exists a continuous solution $q_i = q_i(k^2)$ of Equation (2.1). If the quadratic form (2.3) is positive-definite when $k^2 = k_0^2$ and if the second derivatives occuring in its coefficients are continuous in the quantities q_i, then it remains positive-definite if the second derivatives take on sufficiently small values for any $q_i(k^2)$.

If Π has a minimum, then the steady-state motion is stable [7] and for any arbitrarily small $h, \sigma', n, d_0, \varphi(N)$, we can find such h_0, σ_0', n_0, d_0 that if at the initial instant the inequalities

$$r_0 < h_0, \quad T_{10} < \sigma_0', \quad N_0 < n_0, \quad k^2 - k_0^2 = \delta k^2 < d_0$$
$$\Delta_0 \geqslant \varphi(N_0) \qquad (2.4)$$

are satisfied, then the inequalities

$$r < h, \quad T_1 < \sigma', \quad N < n, \quad \delta k^2 < d_0, \quad \Delta \geqslant \varphi(N) \qquad (2.5)$$

will be satisfied during the whole motion, if during the whole motion the inequality $\Delta \geqslant \varphi(N)$ is not upset. Let us assume that it is not upset. Here T_1 is the kinetic energy of the system relative to the reference system rotating around axis Z with a variable angular velocity $\omega = k/J$.

Let H_0 and H be regions where inequalities (2.4) and (2.5) are fulfilled for a motion corresponding to the value k_0^2, and let H_0' and H' be regions corresponding to the steady-state motion $q_i = q_i\,(k_0^2 + \delta k^2)$. If H_0 is chosen sufficiently small, then any motion starting in the region H_0 can be considered as having started in the region H_0' and can be considered as disturbed around the steady-state motion $q_i\,(k_0^2 + \delta k^2)$ under the condition that $k_0^2 + \delta k^2$ is not disturbed.

Let the system receive a disturbance and let $r', T_1', N', \ldots$ be the same quantities as in (2.4), but taken for the undisturbed motion $q_i\,(k_0^2 + \delta k^2)$ and for the equilibrium surface in this adjacent steady-state motion. Without any change whatsoever in the reasoning presented above, we obtain two Theorems.

Theorem 2.1. If the minimum of Π is found by the quadratic form (2.3), if the region H does not contain the motion of the rigid body and the liquid as a whole, differing from the motion $q_i\,(k_0^2 + \delta k^2)$, and if one of assumptions 1.1 or 1.2 is fulfilled, then $T_1' + r'^2 + N'^2 \to 0$ as $t \to \infty$, and in the case assumption 1.2 is fulfilled, moreover, the squared velocity $\mathbf{V}^2\,(x, y, z, t) \to 0$ as $t \to \infty$.

Theorem 2.2. If with $\delta k^2 = 0$, Π can take negative values, and if with $\delta k^2 = 0$ there is no motion of the rigid body and the liquid as a whole lying entirely in the region H, and if one of assumptions 1.1 or 1.2 is fulfilled, then the steady-state rotation is unstable.

3. Let us now proceed to an analysis of the conditions under which motion of the rigid body and the liquid as a whole is possible.

Let a certain point of the rigid body (a pole) have the accelerations a_x, a_y, a_z along the axes x, y, z and let p', q', r' be the components of the angular velocity of the body with respect to these axes. If the rigid body and the liquid move as a whole, then inside the liquid Equation

$$
\begin{aligned}
&a_x + q''z - r''y + p'\,(p'x + q'y + r'z) - \omega^2 x + \partial W / \partial x = -\,\partial p / \partial x \\
&a_y + r''x - p''z + q'\,(p'x + q'y + r'z) - \omega^2 y + \partial W / \partial y = -\,\partial p / \partial y \\
&a_z + p''y - q''x + r'\,(p'x + q'y + r'z) - \omega^2 y + \partial W / \partial z = -\,\partial p / \partial z
\end{aligned}
\tag{3.1}
$$

must necessarily be satisfied, where $p\,(x, y, z, t)$ is the pressure and $\omega^2 = p'^2 + q'^2 + r'^2$. By assuming that the mixed second derivatives of the pressure with respect to x, y, z are continuous and by equating them, we get $p'' = q'' = r'' = 0$, i.e. the projections of the angular velocity onto the moving axes are constant.

By chosing the axis z along the angular velocity, we get

$$\psi^{\cdot} \sin\theta \sin\varphi + \theta^{\cdot}\cos\varphi = 0, \qquad \psi^{\cdot}\sin\theta \cos\varphi - \theta^{\cdot}\sin\varphi = 0$$

$$\psi^{\cdot}\cos\theta + \varphi^{\cdot} = \omega_0$$

where ψ, φ, θ are Euler angles.

If $\theta(0) \neq 0$ and this can always be assumed, then $\psi^{\cdot} = \theta^{\cdot} = 0$, i.e. the angular velocity vector has a fixed direction in absolute space [8]. By integrating (3.1) we obtain

$$-p = W - {}^1/_2\, \omega^2 (x^2 + y^2) + a_x x + a_y y + a_z z$$

In order that the free surface may remain at rest relative to the body there must have existed such a constant c' that from the condition $p(x, y, z, 0) = c'$ there followed $p(x, y, z, t) = \lambda(t)$ for all x, y, z.

Let us consider the case of a homogeneous field.

Let a homogeneous field act on the liquid and let X_0, Y_0, Z_0 be the coordinates of some point on the rigid body (a pole) in the fixed system, then

$$X - X_0 = x\cos\omega t - y\sin\omega t, \qquad Y - Y_0 = x\sin\omega t + y\cos\omega t$$

$$Z - Z_0 = z$$

if the fixed axis Z is taken parallel to the vector ω.

For a homogeneous field if we take W in the form

$$W = AX + BY + CZ$$

where A, B, C are constants, we obtain

$$\begin{aligned} -p = A\,[x\cos\omega t - y\sin\omega t + X_0] + B\,[x\sin\omega t + y\cos\omega t + Y_0] + \\ + C\,[z + Z_0] - {}^1/_2\,\omega^2 (x^2 + y^2) + [X_0^{\cdot\cdot}\cos\omega t + Y_0^{\cdot\cdot}\sin\omega t] + \\ + [-X_0^{\cdot\cdot}\sin\omega t + Y_0^{\cdot\cdot}\cos\omega t]\, y + Z_0^{\cdot\cdot} z \end{aligned} \tag{3.2}$$

By denoting the initial values of the quantities $A_1, B_1, C_1, A_2, B_2, C_2$ by $X_0, Y_0, Z_0, X_0^{\cdot\cdot}, Y_0^{\cdot\cdot}, Z_0^{\cdot\cdot}$, when $t = 0$ we obtain

$$-p(x, y, z, 0) = (A + A_2)\,x + (B + B_2)\,y + (C + C_2)\,z - {}^1/_2\,\omega^2 (x^2 + y^2)$$

Assuming the origin of the moving system on the free surface we obtain $p_0 = c'$. Hence we have

$$(C + C_2)\,z + p_0 = {}^1/_2\,\omega^2 (x^2 + y^2) - (A + A_2)\,x - (B + B_2)\,y$$

and when $\omega \neq 0$ we obtain, by substituting into (3.2)

$$Z_0 = {}^1/_2\, C_2 t^2 + C't + C''$$

$$(A + X_0^{\cdot\cdot})\cos\omega t + (B + Y_0^{\cdot\cdot})\sin\omega t = A + A_2$$

$$-(A + X_0^{\cdot\cdot})\sin\omega t + (B + Y_0^{\cdot\cdot})\cos\omega t = B + B_2$$

By solving these Equations we have

$$X_0^{\cdot\cdot} = (A + A_2)\cos\omega t - (B + B_2)\sin\omega t - A$$
$$Y_0^{\cdot\cdot} = (A + A_2)\sin\omega t + (B + B_2)\cos\omega t - B$$
$$X_0 = -\omega^{-2}(A + A_2)\cos\omega t + \omega^{-2}(B + B_2)\sin\omega t - {}^1/_2 At^2 + A't + A$$
$$Y_0 = -\omega^{-2}(A + A_2)\sin\omega t - \omega^{-2}(B + B_2)\cos\omega t - {}^1/_2 Bt^2 + B't + B''$$

Using understandable notations these take the form

$$X_0 = \lambda^\circ \sin(\omega t + \varphi_0) - {}^1/_2 At^2 + A't + A''$$
$$Y_0 = \lambda^\circ \cos(\omega t + \varphi_0) - {}^1/_2 Bt^2 + B't + B''$$

If we take the pole on the instantaneous helical axis when $t = 0$ relative to the system having a translational motion in accordance with the law

$$X_0' = -{}^1/_2 At^2 + A't + A'', \qquad Y_0' = -{}^1/_2 Bt^2 + B't + B''$$
$$Z_0' = {}^1/_2 C_2 t^2 + C't + C''$$

(where X_0', Y_0', Z_0' are the coordinates of the origin of the system X', Y', Z' relative to the fixed system), we obtain $\lambda^\circ = 0$.

Hence it follows that in a homogeneous field when $\omega \neq 0$ the pole can move such that in the reference system X', Y', Z' the body rotates with constant angular velocity around the fixed axis.

We note that the field of the external forces and the forces of inertia in this system is parallel to the rotation axis and is homogeneous.

Let $\omega = 0$, then without derogation of generality, we can assume $A = B = 0$. In this case

$$X_0^{\cdot\cdot} = A_2 \frac{C + Z_0^{\cdot\cdot}}{C + C_2}, \qquad Y_0^{\cdot\cdot} = B_2 \frac{C + Z_0^{\cdot\cdot}}{C + C_2}$$

and, moreover, in this case $Z_0^{\cdot\cdot}$ is arbitrary. By integrating we have

$$X_0 = \frac{A_2 C}{C + C_2}\frac{t^2}{2} + A't + A'' + \frac{A_2 Z_0}{C + C_2}$$
$$Y_0 = \frac{B_2 C}{C + C_2}\frac{t^2}{2} + B't + B'' + \frac{B_2 Z_0}{C + C_2}$$

With respect to the system X'', Y'', Z'' translating with axis Z'' parallel to axis Z and with the origin translating relative to the fixed system in accordance with the law

$$X_0'' = \frac{A_2 C}{C + C_2}\frac{t^2}{2} + A't + A'', \qquad Y_0'' = \frac{B_2 C}{C + C_2}\frac{t^2}{2} + B't + B'', \qquad Z_0'' = 0$$

the body translates in a straight line in accordance with any law. It should be noted that the field of the mass forces and the forces of inertia taken from this system, is homogeneous and directed along the straight line on which the body moves. This is correct, of course, only for motions in which tensile stresses do not arise, i.e. $p \geqslant 0$ for all x, y, z inside the liquid and for all $t > 0$.

BIBLIOGRAPHY

1. Krasovskii, N.N., Nekotorye zadachi teorii ustoichivosti dvizheniia (Some Problems in the Theory of Stability of Motion). Fizmatgiz, 1959.
2. Pozharitskii, G.K., Ob asimptoticheskoi ustoichivosti ravnovesii i statsionarnykh dvizhenii mekhanicheskikh sistem s chastichnoi dissipatsiei (On the asymptotic stability of equilibria and of the steady-state motions of mechanical systems with partial dissipation). *PMM* Vol.25, № 4, 1961.
3. Liapunov, A.M., Ob ustoichivosti ellipsoidal'nykh form ravnovesiia vrashchaiushcheisia zhidkosti (On the stability of ellipsoidal forms of equilibria of a rotating liquid). Collected Works, Vol.111,Akad.Nauk, SSSR.
4. Rumiantsev, V.V., Ob ustoichivosti ravnoveiia tverdogo tela s polostiami, chastichno zapolnennymi zhidkost'iu (On the stability of the equilibrium of a solid body with cavities partially filled with a liquid). Dokl. Akad.Nauk, SSSR, Vol.124, № 2, 1959.
5. Mikhlin, S.G., Problema minimuma kvadratichnogo funktsionala (Problem of the Minimum of a Quadratic Functional). Gostekhteoretizdat, 1952.
6. Pozharitskii, G.K. and Rumiantsev, V.V., Zadacha minimuma v voprose ob ustoichivosti dvizheniia tverdogo tela s polost'iu, zapolnennoi zhidkost'iu (Problem of the minimum in the question of stability of motion of a solid body with a liquid filled cavity). *PMM* Vol.27, № 1, 1963.
7. Rumiantsev, V.V., Ob ustoichivosti ustanovivshikhsia dvizhenii tverdykh tel s polostiami, napolnennymi zhidkost'iu (On the stability of steady-state motions of solid bodies with cavities filled with a liquid). *PMM* Vol.26, № 6, 1962.
8. Zhukovskii, N.E., Sobr.soch. (Collected works) ONTI NKTP SSSR 1937.

Translated by N.H.